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NOTE ON INFINITE DETERMINANTS.

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1. Definitions.

Let us consider a doubly infinite series of quantities A_{ik} $(i, k = -\infty, \ldots, +\infty)$, and as is usual with such series we shall suppose the quantities arranged over a plane, so that with any point (i, k) of the plane the quantity A_{ik} is associated; the term A_{00} will be at the origin, and the terms around it will be arranged as follows:

$$\begin{array}{c|ccccccc|c}
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{3,-3} & A_{3,-2} & A_{3,-1} & A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{2,-3} & A_{2,-2} & A_{2\ -1} & A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{1,-3} & A_{1,-2} & A_{1,-1} & A_{1,0} & A_{1,1} & A_{1,2} & A_{1,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{0,-3} & A_{0,-2} & A_{0,-1} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{-1,-3} & A_{-1,-2} & A_{-1,-1} & A_{-1,0} & A_{-1,1} & A_{-1,2} & A_{-1,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{-2,-3} & A_{-2,-2} & A_{-2,-1} & A_{-2,0} & A_{-2,1} & A_{-2,2} & A_{-2,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & A_{-3,-3} & A_{-3,-2} & A_{-3,-1} & A_{-3,0} & A_{-3,1} & A_{-3,2} & A_{-3,3} & \cdots\cdots\cdots \\
\cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots
\end{array}$$

Let us denote by D_1 the determinant indicated by the dotted lines, that is the determinant composed of the quantities A_{ik} $(i, k = -1, 0, +1)$; by D_2 the determinant indicated by the light lines; and by D_3 the determinant indicated by the heavy lines; and in general let us represent by D_m the determinant composed of the quantities A_{ik} $(i, k = -m, \ldots, +m)$, arranged as indicated in the diagram.

Now if as m increases indefinitely, the determinant D_m has a definite limit D, we say that the infinite determinant D_∞ composed of the quantities A_{ik} $(i, k = -\infty, \ldots, +\infty)$ arranged as indicated in the diagram, is convergent, and has D for its value; or, to state the same thing in a form which will be of more use subsequently, the infinite determinant D_∞ is convergent if, for any positive quantity δ given in advance, we can find a positive integer n' such that for any value of n greater than n'

$$| D_{n+p} - D_n | < \delta \tag{1}$$

whatever the value of p. If no such number n' exists, the determinant is divergent.

The *principal diagonal* of the determinant D_∞ is the group of elements arranged along the straight line passing through the origin and making an angle of 45° with the axis of x; it is composed of the quantities A_{ii} $(i = -\infty, \ldots, +\infty)$. The *line* i is composed of the elements grouped along the straight line passing through the point $(i, 0)$, and parallel to the axis of x; similarly the *column* k is composed of the elements grouped along the straight line passing through the point $(0, k)$, and parallel to the axis of y. The element A_{00} is called the *origin* of the determinant; and any element A_{ik} is said to be *diagonal* or *non-diagonal* according as we have $i = k$, or $i \gtrless k$.

Any diagonal element might be taken as the origin, and corresponding to it we would have a new set of determinants D_m; any line of elements parallel to that already chosen might be taken for the principal diagonal, and any element in it might be selected as origin; we thus see that having given a doubly infinite series of quantities, we can form ∞^2 different infinite determinants, and nothing justifies us in stating *à priori* that all of these infinite determinants are convergent or divergent at the same time; or, if they are all convergent, that they have the same value. We thus see that in general an infinite determinant has a perfectly definite meaning only when we know its principal diagonal and its origin.

It is interesting to note that the different infinite determinants considered above are derived from a given arrangement of the terms of the doubly infinite series; to any new arrangement of the terms of this series, there corresponds a new set of ∞^2 different determinants.

2. HISTORICAL SUMMARY.

Infinite determinants were first defined and used by Hill in a paper entitled "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon."* In the discussion of this problem Hill met with an infinite system of linear equations involving an infinite number of unknown quantities; the consideration of such a system of equations naturally suggested the possibility of forming and using an infinite determinant; Hill accordingly defined a convergent determinant, his definition being substantially that given above, and applied the determinants thus defined to the solution of the system of equations under consideration. Thus the discovery of infinite determinants, like that of so many other mathematical functions, was suggested by the consideration of a special problem which demands such functions for its solution.

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* This paper was first published in this country in 1877, and afterwards appeared in Acta Mathematica, t. 8.

Infinite determinants having thus been defined and used in the solution of a special problem, naturally the next and most important step was that of investigating such functions from a purely theoretical standpoint. This was done by Poincaré in a paper entitled "Sur les determinants d'ordre infini."* Poincaré's investigations were limited to determinants satisfying the following conditions: (a) the diagonal elements are all equal to unity, and (b) the sum of the non-diagonal elements is absolutely convergent.

Helge von Koch was the next mathematician to turn his attention to infinite determinants; his results were published in two papers entitled, respectively, "Sur une application des déterminants infinis à la théorie des équations differentielles linéaires," and "Sur les déterminants infinis et les équations differentielles linéaires."† The determinants studied and used by von Koch are more general than those investigated by Poincaré; like the latter, they satisfy condition (b), but condition (a) is generalized so as to include all infinite determinants in which the product of the diagonal elements is absolutely convergent; infinite determinants satisfying these modified conditions are said to be of the *normal form*, and it is seen that the determinants studied by Poincaré form a special class of determinants of the normal form.

In the second paper cited above von Koch shows that a certain class of infinite determinants enjoys the same properties as determinants of the normal form; this class of determinants includes all that satisfy the following conditions: (a) the product of the diagonal elements is absolutely convergent; (b) there exists a series of quantities x_k $(k = -\infty, \ldots, +\infty)$ such that the doubly infinite series

$$\Sigma_i \Sigma_k A_{ik} \frac{x_i}{x_k}$$

is absolutely convergent.

Von Koch, while investigating the subject from a theoretical standpoint, did so for the special purpose of applying his results to a particular problem; and for that reason his results lack that generality which is to be desired. In the introduction to his second paper he states: "Dans ce qui suit nous nous bornerons à développer ce qui paraît nécessaire pour pouvoir appliquer l'instrument nouveau d'une manière absoluement rigoureuse au problème que nous nous sommes proposé."*

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* Bulletin de la Société mathématique de France, t. 14.

† Acta Mathematica, ts. 15, 16.

Craig has also done some work in connection with the application of infinite determinants to differential equations; while his results have been used in his course of lectures on differential equations delivered at Johns Hopkins University, his work has not as yet been published.

In the present paper an attempt is made to place the discussion of infinite determinants upon a slightly new basis, in the hope of giving greater simplicity to the reasoning employed, and at the same time more generality to the theorems established.

3. EXPRESSION OF AN INFINITE DETERMINANT AS AN INFINITE SERIES, THE TERMS OF WHICH ARE INFINITE PRODUCTS.

Returning to the definition of a convergent determinant embodied in the inequality (1), let us push this definition to its legitimate end, and thereby get a clearer view of the problem before us.

We must determine exactly what is meant by saying that the determinants D_m tend towards a limit as m increases indefinitely.

Let us take as a simple illustration the determinant,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 .$$

The configuration of letters, numbers, and lines written on the left hand side of the sign of equality is called a determinant, and is simply a convenient symbol used to represent the algebraic function written on the right hand side; this is the case with all determinants, so that if we speak of a set of determinants D_m as tending towards a limit as m increases indefinitely, we mean that the functions represented by these determinants tend towards a limit as m increases indefinitely.

We must carefully examine the nature of the function represented by a determinant; the function represented by D_m is the algebraic function consisting of $(2m + 1)!$ terms, half of which are positive, and the other half negative, each term consisting of the product of $2m + 1$ of the quantities A_{ik}, every possible combination being taken subject to the condition that one and only one element must be taken from any line (or column) for a given term. Certain conventions are adopted to determine the sign of any term.

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* In the Comptes Rendus for Jan. 21, 1895, there is an article by von Koch entitled, "Sur la convergence des déterminants d'ordre infini et des fractions continues." Unfortunately I have not yet been able to consult this article.

As m increases indefinitely the number of the terms of such a function increases indefinitely, and the number of elements appearing in each term increases indefinitely; so that in the limiting case, an infinite determinant represents an infinite series of terms, some positive and some negative, each term being the product of an infinite number of elements, one and only one being taken from each line (or column) for any term.

In the case of the function represented by a finite determinant, the order of the terms, as well as that of the elements in any term, is a matter of indifference; this is not so with an infinite determinant, however, for in general the value of an infinite sum or an infinite product is dependent upon the order of the terms of the sum, or that of the factors of the product. Consequently, having given an infinite determinant of specified origin and principal diagonal, it is not only necessary to adopt some convention by which to determine the sign of any term, but it is also necessary to adopt conventions that will fix the order of the terms in any determinant and the order of the elements in each term.

Rules sufficient to determine the sign of any term may be formed as an extension of those adopted for finite determinants. In determining the sign of any term the elements are all supposed to be positive; of course any odd number of negative elements in a term would change its sign, but these changes adjust themselves by purely algebraic principles.

The sign of the principal term is determined by the following rule:—

(1) The term $\overset{+\infty}{\underset{-\infty}{\Pi_i}} A_{ii}$, composed of the diagonal elements written in the order in which they occur in the infinite determinant, is positive.

In this term both suffixes occur in their natural order; other terms may be formed from this principal term by permuting either set of suffixes; we shall in all that follows suppose the first set of suffixes to remain fixed, and form the different terms by permuting the second set of suffixes; this amounts to adopting a convention for determining the order of the elements in any term. This being agreed upon, the sign of any term is determined by the following rule:—

(2) Any interchange of two suffixes of the second set, the suffixes of the first set retaining their order, alters the sign of the term.

The convention that the elements in any term be written so that the first set of suffixes are in their natural order is a natural one; in fact it is generally observed in regard to finite determinants, for if we have a determinant the principal term of which is $a_1b_2c_3d_4 \ldots e_n$, in writing any other term it is usual to write the letters in their natural order, the permutations being made among the suffixes; while any violation of this rule in the case of a finite determinant will not alter the value of the determinant, this is not in general true for an infinite determinant and the adoption of such a rule is a necessity.

Having thus adopted a rule which is sufficient to fix the order of the elements in the infinite product of any term, it remains to consider the question of the order of the terms of the infinite series.

Three conventions in regard to the arrangement of the terms of an infinite determinant suggest themselves naturally; they are: (1) to write the principal term first; (2) to choose the order of the terms so that they are alternately positive and negative; (3) to choose the order of the terms so that there is a symmetrical arrangement in regard to the principal term, for example, if the fourth term to the right of the principal term is obtained by interchanging $-m$ and $+n$, the fourth term to the left is to be obtained by interchanging $+m$ and $-n$. Rule (3) is easily seen to be in harmony with (2). Even though we adopt these three conventions, the order of the terms is still undetermined, and there is open to us for choice an infinite number of rules any one of which is sufficient to fix the order of the terms. This being the case, the question naturally arises whether there exists a class of determinants the values of which are independent of the particular law chosen to fix the order of their terms. This question is answered very easily, and furnishes the most natural basis for dividing infinite determinants into two general classes according to the nature of the series represented by the infinite determinant.

However, before proceeding to the consideration of this classification of infinite determinants, it may be well to give a brief résumé of the results arrived at in this article. We have shown that an infinite determinant represents an infinite series, the terms of which are infinite products; we have formulated rules sufficient to determine the sign of any term and the order of the factors in the infinite product constituting any term. We have also seen that any one of an infinite number of possible rules would be sufficient to determine the order of the terms in the series.

4. DISTINCTION BETWEEN THE TWO KINDS OF INFINITE DETERMINANTS.

Before classifying determinants according to the nature of the series represented by them, it is necessary to call attention to a distinction which must be made in regard to the infinite determinants we are to consider. Since the number of the terms of the determinant D_m is $M = (2m + 1)!$, if we have $m = \infty$, M becomes infinite of an order higher than the first, and consequently the function represented by such a determinant is not an ordinary infinite series, but is one of an order higher than the first. This class of infinite determinants for which $m = \infty$, we shall call infinite determinants of the first kind.

However, if instead of considering m as approaching infinity we consider M, the number of terms of the function represented by D_m, as approaching infinity, then for $M = \infty$, the series which the determinant represents is an ordinary infinite series. Since M is finite for any finite value of m, however large, if we have $M = \infty$, m must also be infinite, though necessarily of an order lower than the first. Infinite determinants for which $M = \infty$ we shall call infinite determinants of the second kind. They are infinite determinants in just as strict a sense of the word as are those of the first kind, but in order to be perfectly rigorous it is necessary to make the distinction. In what follows we shall confine our investigations to infinite determinants of the second kind.*

5. SEMI-CONVERGENT AND ABSOLUTELY CONVERGENT DETERMINANTS.

Since the consideration of an infinite determinant has been reduced to the consideration of an infinite series, it is natural to divide infinite determinants into classes according to the nature of the series represented by such determinants.

The first and most important classification of series is that of dividing them into the two classes of convergent and divergent series; and following this classification we may say that an infinite determinant is convergent or divergent according as it represents a convergent or divergent series. [This is not a new definition of convergence, but simply a new way of stating the one already given.]

Convergent determinants may be subdivided into two classes, semi-convergent and absolutely convergent determinants, according as they represent semi-convergent or absolutely convergent series. Every convergent determinant must belong to one of these two classes, and no determinant can belong to both classes.

Absolutely convergent determinants constitute by far the most interesting class of determinants, for the value of the series represented by such a determinant being independent of the order of its terms, we conclude that *an absolutely convergent determinant has the same value, whatever the law adopted to fix the order of the terms in the series represented by it;* and consequently in the study of such determinants we are freed from the necessity of adopting a law to fix the order of the terms of the series. Furthermore since the sign and the order of the factors constituting any term are determined by the rules already given, the only effect introduced by making a change in the choice of principal diagonal or origin, is a change in the order of the terms of the series; hence *the value of an absolutely convergent determinant is independent of the choice of principal diagonal or origin.*

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* While it is necessary to distinguish between the two kinds of infinite determinants, it would seem that there is no loss of generality in confining our investigations to the determinants of the second kind; for in the definition of an infinite determinant m is simply supposed to become indefinitely great, and making M infinite secures this; however, adopting the safer plan, I shall confine my investigations to determinants of the second kind.

Since a semi-convergent determinant represents a semi-convergent series, the value of such a determinant is dependent upon the choice of the law which will fix the order of the terms of the series. If we adopt any rule to determine the order of the terms, we may investigate the conditions of convergence corresponding to that particular rule; but as the number of such rules from which a choice may be made is infinite, it would be useless to investigate the conditions for any one rule, unless it can be shown that for sufficient reasons that rule is to be preferred to all others. It would seem that this is not the case, so that it is not probable that much can ever be done with this class of determinants.

Since a change in the origin must introduce a change in the order of the terms of the series, it follows that *a semi-convergent determinant is dependent upon the choice of its origin for its value.*

6. THEOREMS ON CONVERGENT DETERMINANTS.

Following the method generally adopted in the treatment of infinite series, we shall prove certain theorems concerning convergent determinants, reserving for the present the investigation of the question of convergence. Considerable generality can be given to such theorems owing to the fact that their proof rests upon the proposition that a convergent infinite determinant can be represented by a convergent infinite series, the terms of which are arranged according to some definite law; a little reflection will show that the proof is independent of the particular law that might be chosen.

THEOREM I. An absolutely convergent determinant remains absolutely convergent if we replace the elements of any line (or column) by a series of quantities all of which are less in absolute value than a given positive quantity.

Let us represent the infinite determinant D_∞ by the expression

$$\sum_{i\,-\infty}^{+\infty} B_i\,, \tag{1}$$

where B_i represents the product of an infinite number of factors A_{ik}, the sign of any term and the order of the factors in that term being governed by the rules already adopted, and the order of the terms of the series being fixed by some law which we shall suppose known.

To fix the idea let us suppose the elements of the line 0

$$\ldots, A_{0,-m}, \ldots, A_{00}, \ldots, A_{0m}, \ldots$$

to be replaced by the quantities

$$\ldots, s_{-m}, \ldots, s_0, \ldots, s_m, \ldots$$

satisfying the inequalities

$$|s_i| < k. \qquad k > 0.$$

By hypothesis the series

$$\sum_{i,\,-\infty}^{+\infty} |B_i|$$

is convergent; consequently the series

$$\sum_{i,\,-\infty}^{+\infty} k\,|B_i'|, \tag{2}$$

where B_i represents B_i divided by its factor from the line 0, is convergent; the series

$$\sum_{i,\,-\infty}^{+\infty} |C_i|, \tag{3}$$

where C_i represents B_i' multiplied by the s corresponding to its element from the line 0, is also convergent since its terms are respectively equal to or less than the corresponding terms of the series (2). Since the series (3) is convergent, the determinant which represents

$$\sum_{i,\,-\infty}^{+\infty} C_i$$

is absolutely convergent, and the theorem is proved.

COR. 1. An absolutely convergent determinant remains absolutely convergent if we replace any finite number of lines and columns by series of quantities all of which are less in absolute value than a given positive quantity.

This proposition regarding infinite determinants is the analogue of the proposition that in considering the convergence of a series we may neglect any finite number of terms. This theorem was first proved by Poincaré for the class of determinants which he investigated, and was afterwards extended by von Koch to determinants of the normal form.

COR. 2. A semi-convergent determinant remains semi-convergent if we replace the elements of any finite number of lines and columns by series of quantities all of which are less than a given positive quantity, and each of which has the same sign as the element that it replaces.

The proof of this proposition is the same as that of the theorem for absolutely convergent determinants; the necessity for making the restriction that the quantities have the same sign as the elements they replace is made clear by considering the case in which the terms of the series representing the semi-convergent determinant are some negative and the others positive; the signs of the s's might be such as to make the terms of the series represented by the new determinant all positive; since the original determinant is only semi-convergent, the series thus formed is divergent, and consequently the determinant has lost its convergence by replacing its elements by the s's. If however the s's all have the same signs as the elements they replace, the signs of the terms of the series cannot be changed, and the determinant remains semi-convergent when we replace the elements of a given line by the s's.

COR. 3. A determinant (of the second kind) which is of the normal form is an absolutely convergent determinant.

Every convergent determinant of the second kind must belong to one of the two classes, semi-convergent or absolutely convergent determinants. Von Koch has proved that a determinant of the normal form is convergent, and that if in such a determinant we replace the elements of any line or column by any series of quantities, positive or negative, all of which are less in absolute value than a given positive quantity, the determinant does not lose its convergence; it follows from the preceding theorem that this is the case *without restriction* only for absolutely convergent determinants; hence determinants of the normal form (when of the second kind) come under the class of absolutely convergent determinants, and any proposition proved for the latter is also true for the former.

THEOREM II. If in an absolutely convergent determinant we interchange any two lines (or columns), the sign of the determinant is changed; if any two lines (or columns) are identical the determinant vanishes.

Since the determinant is convergent, we have

$$| D_{m+p} - D_m | < \delta . \tag{1}$$

If we denote by D'_m, the value of the determinant D_m when the two lines (or columns) are interchanged, we have by the preceding theorem

$$| D'_{m+p} - D'_m | < \delta . \tag{2}$$

Now m can always be chosen so large that the two lines (or columns) interchanged will be included in D_m; hence

$$D'_m = - D_m . \tag{3}$$

But we have

$$| D_{m+p} - D_m + D'_{m+p} - D'_m | \leqq 2\delta \,, \tag{4}$$

or in consequence of (3)

$$| D_{m+p} - D'_{m+p} | \leqq 2\delta \,; \tag{5}$$

hence if we represent by D and D' the limits towards which D_m and D'_m tend as m increases indefinitely, we have for m infinitely great

$$D = - D'. \tag{6}$$

The second part of the theorem follows immediately; if two lines (or columns) are identical, the value of the determinant can not be altered by interchanging them; but the sign of the determinant must be altered by this interchange, hence the determinant vanishes.

COR. If a semi-convergent determinant does not lose its convergence when we interchange two lines (or columns) the sign of the determinant is changed; if any two lines (or columns) are identical, the determinant vanishes.

The proof of this proposition is the same as that for absolutely convergent determinants.

THEOREM III. If every element of any line (or column) of a convergent determinant be multiplied by the same factor k, the determinant is multiplied by that factor.

Since every term of the series

$$\sum_{-\infty}^{+\infty}{}_i B_i \tag{1}$$

is a convergent product, and contains one and only one element from any line (or column), each term of the series is multiplied by the factor k; and since the series is convergent, the series formed by multiplying each term by the same factor is convergent, and has for its value

$$k \sum_{-\infty}^{+\infty}{}_i B_i \,; \tag{2}$$

hence if the value of the original determinant is D, the value of the new determinant is kD.

COR. 1. If the elements of any line (or column) are the same multiples of the corresponding elements of some other line (or column) the determinant vanishes.

By the theorem just given we can divide the elements of that line (or column) by the common factor; two lines (or columns) being then identical, the determinant vanishes.

Cor. 2. If the signs of all the elements of any line (or column) be changed, the sign of the determinant is changed.

Such a change of sign of the elements is equivalent to multiplying these elements, and consequently the determinant, by the factor -1.

Theorem IV. If each element of any line (or column) of an absolutely convergent determinant can be resolved into the sum of n quantities, the given determinant can be expressed as the sum of n absolutely convergent determinants.

To fix the idea, let us suppose that the elements of the line 0 can be resolved into the sum of n quantities, so that we have

$$A_{0m} = a_{1m} + a_{2m} + a_{3m} + \dots + a_{nm} .$$

Owing to the absolute convergence of the series

$$\sum_{-\infty}^{+\infty}{}_i\, B_i ,$$

each of the series

$$\sum_{-\infty}^{+\infty}{}_i\, a_{ri} B'_i \qquad (r = 1, 2, \dots, n.)$$

is absolutely convergent; hence we can write

$$\begin{aligned} \sum_{-\infty}^{+\infty}{}_i\, B_i &= \sum_{-\infty}^{+\infty}{}_i\, (a_{1i} + a_{2i} + \dots + a_{ni})\, B'_i \\ &= \sum_{-\infty}^{+\infty}{}_i\, a_{1i} B'_i + \sum_{-\infty}^{+\infty}{}_i\, a_{2i} B'_i + \dots + \sum_{-\infty}^{+\infty}{}_i\, a_{ni} B'_i ; \end{aligned}$$

or

$$D = \Delta_1 + \Delta_2 + \dots + \Delta_n ,$$

where D is the original determinant, and Δ_r is the determinant formed by replacing the elements of the line 0 of the original determinant by the quantities $a_{ri}\ (i = -\infty, \dots, +\infty)$. By Theorem I each of these determinants is absolutely convergent, and the theorem under consideration is thus proved.

Cor. The theorem holds for a semi-convergent determinant, if all the elements of the line 0 in each of the determinants Δ_r have the same sign as the corresponding elements in the determinant D.

This follows as a consequence of Theorem I, Cor. 2.

7. TESTS OF CONVERGENCE.

Before attempting a general discussion of the conditions of convergence for infinite determinants, we shall give two theorems concerning determinants of special form, which may be interesting in themselves, as well as of use subsequently.

THEOREM I. An infinite determinant all the terms of which are negative is divergent.

Any term of the series

$$\sum_{i}{}_{-\infty}^{+\infty} B_i$$

consists of the product of an infinite number of negative factors; the product of an even number of these factors gives a positive quantity; the introduction of another factor changes the sign of the product, and as we thus increase the number of the factors, the sign of the product continually changes from plus to minus; the product is therefore divergent, and consequently the determinant is divergent.

THEOREM II. In order that an infinite determinant, all the terms of which are positive, may be convergent, it is necessary and sufficient that the terms of the series

$$\sum_{i}{}_{-\infty}^{+\infty} B_i$$

tend to zero as a limit.

Since the elements of the determinant are all positive, it will follow as a consequence of the law governing the sign of any term, that half of the terms of the series

$$\sum_{i}{}_{-\infty}^{+\infty} B_i$$

are positive and the other half negative; in whatever manner they may be grouped only a finite number of positive or negative terms can fall together; since these terms by hypothesis tend to zero as a limit, the sum of any finite number of terms will also tend to zero as a limit; the series has thus been reduced to one in which the terms are alternately positive and negative, and tend to zero as a limit; such a series is known to be convergent, and the theorem is proved.

The consideration of these two theorems places us in a better position for formulating the necessary conditions that an infinite determinant be convergent. In the first place in order that an infinite determinant which is represented by the series

$$\sum_{i}{}_{-\infty}^{+\infty} B_i$$

be convergent, it is necessary that each one of the infinite products B_i be convergent. Such an infinite product may become divergent in one of two ways; it may either tend to infinity as a limit, or its sign may be alternately positive and negative owing to the repeated introduction of a negative factor.

Let us write $A_{ik} = \pm (1 + a_{ik})$; if any of the quantities A_{ik} are less in absolute value than unity, the corresponding a_{ik} is to be taken as equal to zero. Let us now form the series

$$\Sigma a_{ik} ,$$

where the quantities a_{ik} are selected and arranged according to the same rules as the quantities A_{ik} in the products B_i; there will be an infinite number of these series, one corresponding to each product B_i; let us represent these series by S_i ($i = -\infty , \ldots , +\infty$). In order that any product B_i be convergent it is necessary that the corresponding series S_i shall be convergent; hence we are able to state as a first necessary condition that the infinite determinant be convergent, that each one of the series S_i ($i = -\infty , \ldots , +\infty$) must be convergent.

Any one of the terms B_i will be divergent if it contains an infinite number of negative factors which are in absolute value equal to or greater than unity. An infinite number of negative factors all less in absolute value than unity would not make it divergent, for while its sign would be alternately positive and negative, it would tend to zero as a limit; hence we may state as a second necessary condition that any term B_i must contain only a finite number of negative factors satisfying the inequality

$$| A_{ik} | \geqq 1 .$$

Supposing that the convergence of each term of the series

$$\sum_{-\infty}^{+\infty}{}_i \, B_i$$

is secured by the fulfilling of the two conditions already given, in order that the infinite series may be convergent it is necessary that its terms tend to zero as a limit; that is, we must always be able to find an integer n', such that for any value of n greater than n'

$$| B_{\pm n} | < \delta ,$$

δ being any positive quantity which may be chosen as small as we please; in order that this inequality may exist, each term $B_{\pm n}$ ($n > n'$) must either contain a factor which is evanescent, or it must contain an infinite number of factors each less than unity.

To give a résumé of the results arrived at, we may state that in order that an infinite determinant may be convergent, the following conditions are necessary:—

1. Each of the series S_i ($i = -\infty , \ldots , +\infty$) must be convergent.

2. Any term must contain only a finite number of negative factors satisfying the inequality

$$|A_{ik}| \geqq 1\,.$$

3. Each term after a certain term in the series is reached must either contain an evanescent factor or an infinite number of factors each of which is less than unity.

While these are necessary conditions, they are not in general sufficient to determine the convergence of a determinant; however, in connection with the preceding theorems they enable us to state that certain classes of determinants are convergent; for example, any determinant satisfying the following conditions is convergent: (a) the elements are all positive; (b) each of the series $S_i\,(i = -\infty, \ldots, +\infty)$ is convergent; (c) for some line i (or column k) the elements A_{ik} tend to zero as we go farther from the origin.

Also any determinant satisfying the following conditions is convergent: (a) the elements are all positive; (b) each of the series $S_i\,(i = -\infty, \ldots, +\infty)$ is convergent; (c) an infinitely great number of lines (or columns) are composed of quantities less in value than unity.

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